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On Two Conjectures of Stanton and Mullin

J. I. HALL

Department of Mathematics, Eindhoven University of Technology, Eindhoven, The Netherlands

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Following Stanton and Mullin [9] an (r, λ)-*system* is a set V of v varieties and a collection of b nonempty subsets of V called blocks such that every variety occurs in precisely r blocks and every pair of varieties occurs in exactly λ blocks. We also have the nondegeneracy condition $r > \lambda \geqslant 0$. Thus an (r, λ)-system is a pairwise balanced design of index λ with constant replication number r (cf. [2, 10]). In particular, any block design (cf. [4]) is an (r, λ)-system where the parameters r and λ have their usual interpretation.

For any (r, λ)-system we define

$$R(r, \lambda) := [r(r-1)/\lambda] + 1.$$

It is easily seen that, for a block design, the inequality

$$v \leqslant R(r, \lambda) \tag{0.1}$$

is equivalent to Fisher's inequality [4, (10.2.3)]. For (r, λ)-systems in general the inequality (0.1) is not valid. We call an (r, λ)-system C^* *reducible* if it has a block containing all varieties of V (a *complete block*) or a collection of v one-element blocks whose union is V. If C^* is not reducible, it is *irreducible*. Stanton and Mullin [9] make the following two conjectures.

Conjecture 1. For $\lambda \leqslant 2$ (and perhaps all λ), $v = R(r, \lambda)$ implies $v = b$ if the corresponding (r, λ)-system is irreducible.

Conjecture 2. For $\lambda \leqslant 2$ (and perhaps all λ), an (r, λ)-system with $v > R(r, \lambda)$ is reducible.

As Stanton and Mullin point out, the conjectures can be directly verified for $\lambda = 1$. In this note we shall discuss these two conjectures and related problems. In particular we shall resolve the conjectures by proving Conjecture 2 for $\lambda = 2$ and displaying counterexamples to both conjectures for all other suitable λ at least 2. We also give a description of $(r, 2)$-systems satisfying (0.1) with equality. Counterexamples to Conjecture 1 have been given by Mukhapadhyay [5] and De Witte [11] for $\lambda = 2$, Ervynck for $\lambda = 3$ [11], and Bhagwandas and Sastry [1] for all $\lambda \geqslant 2$ when there exists an Hadamard matrix of side 4λ.

With each (r, λ)-system C^* we associate its (0, 1) point–block incidence matrix C, a v by b matrix which satisfies

$$CC^T = (r - \lambda)I + \lambda J, \tag{0.2}$$

I the v by v identity matrix and J the v by v matrix all of whose entries are 1. In particular, as $r > \lambda$ the right-hand side of (0.2) is nonsingular. Thus we have the Fisher-type inequality

$$v \leqslant b \tag{0.3}$$

for (r, λ)-systems. Conversely, any v by b (0, 1) matrix satisfying (0.2) can be considered as the incidence matrix of an (r, λ)-system C^*. A result of Ryser [8] shows that (r, λ)-systems, such as those mentioned in Conjecture 1, which have $v = b$ and so meet the bound (0.3) must have constant block size r and so are symmetric designs with parameters (v, r, λ).

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Notation. We now define certain (0, 1) matrices for use throughout the paper:

I_p is the p by p identity matrix;

$0_{p,q}$ is the p by q matrix with all entries 0;

$J_{p,q}$ is the p by q matrix with all entries 1;

$F^{(i)}_{p,q}$ is the p by q matrix with 1's in row i and 0's elsewhere;

$G^{(i)}_{p,q}$ is the p by q matrix with 0's in column i and 1's elsewhere.

If the size of the matrix is apparent, we may leave out the indices.

1. A Construction for Certain (r, λ)-systems with $\lambda \geqslant 3$

In this section we shall construct counterexamples to Conjectures 1 and 2 for all λ at least 3. We shall say that an (r, λ)-system C^* and its incidence matrix C have property I if

$$v = R(r, \lambda) \qquad \text{and} \qquad b > v, \tag{I}$$

and that C^* and C have property II if

$$v > R(r, \lambda). \tag{II}$$

Thus Conjectures 1 and 2 maintain that, for all λ, an (r, λ)-system having property I or property II is reducible.

We shall call an (r, λ)-system C^* in which all block sizes are at least 2 and not more than $v - 2$ *strongly irreducible*. Otherwise C^* is *strongly reducible*. In particular, a strongly irreducible (r, λ)-system is irreducible, and a reducible (r, λ)-system is strongly reducible. If C is the incidence matrix of a strongly irreducible (r, λ)-system, then each column sum c of C satisfies $2 \leqslant c \leqslant v - 2$. Hence we shall call any (0, 1) matrix C with v rows *strongly irreducible* if each of its column sums c satisfies $2 \leqslant c \leqslant v - 2$. Otherwise, C is *strongly reducible*. The (r, λ)-systems constructed by Mukhapadhyay [5], Bhagwandas and Sastry [1], and De Witte [11] have blocks of size 1 or $v - 1$ and so are strongly reducible. For any (r, λ)-system C^* we define $k := r - \lambda$, the *order* of C^*.

PROPOSITION 1. *For all $\lambda \geqslant 3$, there exists an (r, λ)-system $D^*(r, \lambda)$ which has property* I *and is strongly irreducible.*

PROPOSITION 2. *For all $\lambda \geqslant 3$, there exists an (r, λ)-system $C^*(r, \lambda)$ which has property* II *and is strongly irreducible.*

Suppose C^* is a $(k + 1, 1)$-system containing a *parallel class* of blocks, that is, a set of disjoint blocks whose union contains all varieties of C^*. In terms of the point–block incidence matrix C of C^*, this says that C has a set of columns with the property that every row of C has precisely one 1 in the columns of the set. Suppose further that $\mathbf{x}$ is the (0, 1) row vector of length b whose ones are in the positions corresponding to the blocks (columns of C) of the parallel class. Considering $\mathbf{x}$ and the rows of C as vectors from a space of dimension b over $GF(2)$, we let D be the v by b matrix whose rows are $\mathbf{x} + \mathbf{c}$, as $\mathbf{c}$ runs through the rows of C. We observe that

$$DD^T = kI + (m - 1)J,$$

where m is the number of blocks in the parallel class. Checking (0.2) we see that D is an incidence matrix for a $(k + m - 1, m - 1)$-system D^* which has the same number of varieties v and blocks b as C^*. We call D^* the *complement* of C^* with respect to the given parallel class, since D^* is the system obtained by replacing in C^* the blocks of the class by their complements. Note that if the block sizes for C^* are $\{b_i \mid i = 1,..., b\}$ then those for D^* are from $\{b_i, v - b_i \mid i = 1,..., b\}$. In particular, D^* is strongly irreducible if and only if C^* is strongly irreducible.

We give two examples of this construction. Let P^* be the projective plane of order 5. We delete a point x from P^* to obtain a (6, 1)-system $P_x{}^*$ with $v = 30$ and $b = 31$. $P_x{}^*$ has a parallel class of size 6, being those blocks which originally contained x. We let $C^*(10, 5)$ be the complement of P_x^* with respect to that class, so that $C^*(10, 5)$ is a $(10, 5)$-system with $v = 30$ and $b = 31$. $C^*(10, 5)$ is strongly irreducible as all block sizes of P_x^* are 5 or 6.

Next suppose we delete from P^* a set X of 12 points of P^* with the property that no five points of X are on a single line of P^*. The resulting $(6, 1)$-system P_X^* has $v = 19$, $b = 31$, and as before several parallel classes of size 6. Forming the complement $D^*(10, 5)$ with respect to one of these classes, we see that D^* is a $(10, 5)$-system with $v = 19$ and $b = 31$. Again, D^* is strongly irreducible.

Note that $R(10, 5) = 19$. Hence $D^*(10, 5)$ has property I and $C^*(10, 5)$ has property II. Thus we have in particular displayed the two propositions for $\lambda = 5$.

A *transversal design* with block size 4, $TD(4, k)$, is a $(k + 1, 1)$-system with $v = 4k$ which contains a parallel class of four blocks of size k and all other blocks of size 4. It is not difficult to see that each block of size 4 intersects each block of size k and that $b = k^2 + 4$. It is well known (cf. [4]) that the existence of a $TD(4, k)$ is equivalent to the existence of a pair of orthogonal Latin squares of order k. In particular, by a result of Bose, Shrikhande, and Parker [7], $TD(4, k)$ exist for all positive integers k not 2 or 6. The following lemma is of use in proving Propositions 1 and 2.

LEMMA 1.1. *If k is a positive integer not 2, 3, or 6, then there exists a $TD(4, k)$ with a parallel class of k blocks of size 4.*

Proof. Such a parallel class corresponds to a common transversal of the corresponding Latin squares. A pair of squares of order 10 with this property was first given by Weisner [14]. For $k \neq 10$, the result is attributed by Hedayat, Parker, and Federer [13] to Hedayat and Seiden, although we have not been able to check completely this reference. In any case, the result is an immediate corollary to the theorem of Brayton, Coppersmith, and Hoffman [12] on Latin squares orthogonal to their transpose.

Suppose $k \geqslant 4$ and $k \neq 6$; and let P^* be a transversal design $TD(4, k)$ with a parallel class of k blocks of size 4, as in Lemma 1.1. We now let C^* be the complement of P^* with respect to the parallel class. Thus C^* is a $(2k - 1, k - 1)$-system with $v = 4k$ and $b = k^2 + 4$ whose block sizes are from $\{4, k, 4k - 4\}$. If we further delete a point from C^*, we gain a $(2k - 1, k - 1)$-system D^* with $v = 4k - 1$ and $b = k^2 + 4$ whose block sizes are from $\{3, 4, k - 1, k, 4k - 5, 4k - 4\}$.

Noting that $R(2k - 1, k - 1) = 4k - 1$, we see that C^* has property II and is strongly irreducible while D^* has property I and is strongly irreducible. Taking $C^*(2k - 1, k - 1) := C^*$ and $D^*(2k - 1, k - 1) := D^*$, Propositions 1 and 2 are true for $\lambda = k - 1$. By Lemma 1.1 this displays both propositions for λ not 5. For $\lambda = 5$, the systems $D^*(10, 5)$ and $C^*(10, 5)$ constructed above give the propositions.

It is clear that numerous $(k + 1, 1)$-systems other than the $TD(4, k)$ could be chosen to construct $(2k, k)$-systems and $(2k - 1, k - 1)$-systems with many more points than $R(2k, k) = R(2k - 1, k - 1) = 4k - 1$. We remark that the constructions given here and others similar to them can be used again to prove both propositions with the inequality $\lambda \geqslant 3$ replaced by $k = r - \lambda \geqslant 3$. Similarly in Proposition 2 we could replace $\lambda \geqslant 3$ by $r \geqslant 6$. In Proposition 1, $\lambda \geqslant 3$ can be replaced by $r \geqslant 6$ with the possible exception of $r = 11$.

Construction of strongly irreducible (r, λ)-systems with property II and $r \notin \{2\lambda, 2\lambda + 1\}$ can be done in a similar way to that done here. The corresponding constructions for (r, λ)-systems with property I are not so obvious, since $R(r, \lambda)$ is frequently not an integer. For instance, a counterexample to Conjecture 1 with $r = 11$ must have $\lambda = 5$.

2. Maximal $(r, 2)$-systems

In this section we show that an $(r, 2)$-system without a complete block has at most $\binom{r}{2} + 1$ $(= R(r, 2))$ varieties. We describe all such $(r, 2)$-systems which meet this bound.

Consider first a $(4, 2)$-system C^* with 8 blocks. Thus the incidence matrix C of C^* has rows of length 8 containing 4 ones and having all inner products 2. If we replace any collection of rows of C by their complements (replacing 0's by 1's and 1's by 0's) the new matrix C_1 still satisfies $C_1 C_1^T = 2I + 2J$. Therefore, if C_1 has no zero columns, it is the incidence matrix of a new $(4, 2)$-system C_1^*. We call C^* and C_1^* *Hadamard equivalent*. If we replace rows of C in such a way as to construct in C_1 a column of 1's we can delete that column to gain a matrix P with 7 columns such that $PP^T = 2I + J$. By (0.3) P (and hence C) has at most 7 $(= R(4, 2))$ rows. Such a P with 7 rows must be an incidence matrix for the projective plane of order 2. We thus see that a $(4, 2)$-system C^* with 8 blocks has at most 7 varieties, this number being achieved only if C^* is Hadamard equivalent to the union of the projective plane of order 2 and a complete block. It is not hard to see that there are two such nonisomorphic irreducible C^* and that both are strongly reducible (see [11]).

Lemma 2.1. *Let C^* be a $(4, 2)$-system with at most one complete block and $v \geqslant 7$. Then $v = 7$ and either*

(i) $b = 7$ *and C^* is a symmetric $(7, 4, 2)$-design, or*

(ii) $b = 8$ *and C^* is Hadamard equivalent to the union of the projective plane of order 2 and a complete block.*

Proof. Every two distinct rows of the incidence matrix C have inner product 2. Suppose the rows and columns of C cannot be permuted so that

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ & & & & \vdots & & & & \end{pmatrix}. \tag{2.1}$$

Then, considering columns 1 through 4, we see that $7 \leqslant v \leqslant \binom{4}{2} + 1 = 7$ and all column sums of C are 4. Hence C^* is a symmetric (7, 4, 2)-design as in (i). Thus we may assume that C has the form (2.1), hence $b \geqslant 8$. As C has at most one column constantly 1, we now find that in fact $b = 8$. We have seen above that in this case C^* must be as in (ii).

We remark that if we momentarily allow empty blocks, then the configurations of Lemma 2.1(ii) include the union of the design of Lemma 2.1(ii) with an empty block.

If we take any symmetric $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$-design with $\lambda \geqslant 2$, an Hadamard design, then by adding to it a complete block we have a reducible $(2\lambda, \lambda)$-system C^* with property I, as $b = v + 1 = 4\lambda$. In a similar manner to the case $\lambda = 2$ considered above, we may complement rows of C to obtain new, usually irreducible $(2\lambda, \lambda)$-systems with the same parameters. These new systems we shall again call *Hadamard equivalent* to C^*. Conversely, it is easily seen that any $(2\lambda, \lambda)$-system with $b = v + 1 = 4\lambda$ is Hadamard equivalent to the union of some symmetric $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$-design and a complete block. In particular, all the $(2\lambda, \lambda)$-systems of Mukhapadhyay [5], Bhagwandas and Sastry [1], and De Witte and Ervynck [11] are of this type. The equivalence classes contain strongly irreducible systems if and only if $\lambda > 2$.

Propositions 1 and 2 and Lemma 2.1 show that Conjecture 1 is false for all λ at least 2 while Conjecture 2 is false as long as $\lambda \geqslant 3$. The following theorem will thus resolve the conjectures by proving that Conjecture 2 is true for $\lambda = 2$.

THEOREM. *Let C^* be an $(r, 2)$-system with $v \geqslant \binom{r}{2} + 1$. If C^* has no complete block, then $v = \binom{r}{2} + 1$ and either*

(i) *C^* is a symmetric $(v, r, 2)$-design,*

(ii) *C^* has a block of size $v - 1$ and r is even, or*

(iii) *$r = 4$, $v = 7$, $b = 8$, and C^* is Hadamard equivalent to the union of a projective plane of order 2 and a complete block.*

Proof. If all block sizes of C^* are equal to r, then C^* is a symmetric $(v, r, 2)$-design as in (i). Hence we assume that some block size is not r. By Lemma 2.1, if $r = 4$ then (i) or (iii) holds. For $r = 3$ it is easily seen that (i) holds. When $r = 5$, if we are not in case (i) then we may permute rows and columns of the incidence matrix C so that

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\ & & & & & \vdots & & & & & & \end{pmatrix}.$$

The arguments which then contradict $v \geqslant 11$ are omitted. We now assume $r \geqslant 6$.

We let C be the incidence matrix of C^* and $v = \binom{r}{2} + 1 + \epsilon$, defining the nonnegative integer ϵ. The rows of the matrix M given by

$$M := \begin{pmatrix} 0_{1,r-4} & 0_{1,b} \\ J_{v,r-4} & C \end{pmatrix}$$

form the words of an equidistant $(v + 1, 2(r - 2), b + r - 4)$-code, in the sense of Deza [3]. By Lemma 3.1 of that paper

$$c(v + 1 - c) \leqslant (r - 2)(v + 1), \tag{2.2}$$

where c is any column sum of M. We claim that each column sum of C is either less than $r + 1$ or more than $v - r$. For otherwise the inequality (2.2) is valid for $c = r + 1$. Substituting for c and v and simplifying, we find

$$r^2 - 7r + 10 + 6\epsilon \leqslant 0,$$

which contradicts $r \geqslant 6$. Thus for any column sum c of C either

$$c \leqslant r \qquad \text{or} \qquad c \geqslant v + 1 - r.$$

In the first instance we say the column is *light* and in the second *heavy*.

Suppose now that the 1's of the first row of C are in the initial r columns and that column i has sum c_i. Then

$$\sum_{i=1}^{r} c_i = r + 2(v - 1) = r^2 + 2\epsilon.$$

Thus the average column sum over the 1's of any given row is $r + (2\epsilon/r)$. As by assumption not all column sums of C are r, some column sum c satisfies $c > r$. This column must then be heavy and satisfy $c \geqslant v + 1 - r$. In fact, in the case $\epsilon > 0$ this argument shows that every row of C has a 1 in a heavy column. As C has no column constantly equal to 1, this in turn would imply that C had at least two heavy columns.

We first assume that C has two or more heavy columns and permute rows and columns of C so that columns 1 and 2 are heavy and

$$C = \begin{pmatrix} J_{s,2} & F^{(1)}_{s,r-2} & \cdots & F^{(s)}_{s,r-2} & 0 \\ G^{(1)}_{t,2} & & & & \\ & T_1 & \cdots & T_s & T_{s+1} \\ G^{(2)}_{u,2} & & & & \\ O_{v,2} & U_1 & \cdots & U_s & U_{s+1} \end{pmatrix}.$$

Each of the middle $t + u$ rows must have one 1 in each matrix T_i for $1 \leqslant i \leqslant s$. Since the first two columns are heavy, $1 \leqslant t + v \leqslant r - 1$ and $1 \leqslant u + v \leqslant r - 1$; hence

$$v - 2(r - 1) \leqslant s \leqslant r - 1.$$

On substituting for v,

$$r^2 - 7r + 8 + 2\epsilon \leqslant 0,$$

which contradicts $r \geqslant 6$. Therefore we may assume that C has precisely one heavy column. In particular, $\epsilon = 0$.

We now permute rows and columns so that

$$C = \begin{pmatrix} J_{s,1} & P \\ 0_{t,1} & Q \end{pmatrix},$$

where $1 \leqslant t \leqslant r - 1$ and $s + t = \binom{r}{2} + 1$. Choose a row, $\mathbf{x}$ say, of Q. Pick a column of C with a 0 in $\mathbf{x}$, and suppose its intersection with P has sum p. The p rows of P with 1's in our chosen column have inner product 1 with each other and 2 with $\mathbf{x}$. Therefore $p \leqslant r/2$. Thus we see that any 0 of Q lies in a column of C whose intersection with P has sum at most $r/2$.

Suppose now that $t \geqslant 2$. Letting $\mathbf{y}$ be a second row of Q, $\mathbf{x}$ and $\mathbf{y}$ have inner product 2. Thus at most two columns of P which correspond to the 1's of $\mathbf{x}$ can have sums greater than $r/2$, and as these columns are light they have sum at most $r - 2$. Counting inner products of $\mathbf{x}$ with the s rows of P, we have

$$2s \leqslant (r - 2)(r/2) + 2(r - 2).$$

As $s \geqslant v + 1 - r$, this leads to

$$r^2 - 8r + 16 \leqslant 0.$$

This again contradicts $r \geqslant 6$. Hence $t = 1$ and $s = v - 1$. Now considering the inner products of a row of P with all other rows of P

$$\binom{r}{2} - 1 \leqslant 2(r - 2) + (r - 3)[(r/2) - 1].$$

Equality holds; so r is even, as in (ii) of the theorem.

In view of previous remarks and the theorem, we have

COROLLARY. *A strongly irreducible $(r, 2)$-system with $v \geqslant \binom{r}{2} + 1$ is a symmetric $(v, r, 2)$-design.*

We observe that numerous examples of case (ii) of the theorem exist. To see this, first choose a projective plane of order k which contains a complete oval ($k + 2$ points, no three on a line). All Desarguesian planes of even order contain complete ovals. We let P be the line–point (0, 1) incidence matrix of all lines from the plane meeting the oval twice and $\mathbf{x}$ the incidence vector of the oval. Now we take

$$C := \begin{pmatrix} J_{s,1} & P \\ 0 & \mathbf{x} \end{pmatrix}$$

where $s := \binom{k+2}{2}$. Then we have

$$CC^T = kI + 2J,$$

and C is the incidence matrix of a $(k + 2, 2)$-system C^* with $v = \binom{k+2}{2} + 1$ and $b = k^2 + k + 2$. C^* has property I and is irreducible. In fact, although never strongly irreducible, any system as in (ii) of the theorem will have property I and be irreducible. All examples of (ii) known to us are of the type described.

For larger λ than 2, some of the methods of this section apply. For a given λ and all suitably large r, (2.2) may be used to show that (r, λ)-systems with $v \geqslant R(r, \lambda)$ are either symmetric (v, r, λ)-designs or have large blocks. It is already known [3, 6] that, for k "small" compared to λ, all $(k + \lambda, \lambda)$-systems with property II are reducible. It is conceivable that for each λ there exists an $r(\lambda)$ such that, for all $r \geqslant r(\lambda)$, (r, λ)-systems with property II are reducible.

For $\lambda = 3$, no irreducible (r, λ)-systems with property II exist with $r = 4$ or 5 while examples for $r = 6$, 7, and 8 can be constructed from the projective planes of order 3, 4, and 5, respectively. We conjecture that for $r \geqslant 9$ such systems are reducible and have been able to prove this for $r \geqslant 21$.

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